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THE

ASSURANCE MAGAZINE,

AND

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Determination of the Rate of Interest of an Annuity.

By PROFESSOR DE MORGAN.

THIS subject, as it left the hands of Simpson, Halley, Dodson, &c., is fully discussed in the appendices of Francis Baily's work on interest and annuities. These, with Milne's note on the subject (vol. ii., p. 680), Jones's notice of the problem, and the various modes of approximation announced in the pages of this *Journal*, give all I have found written about the matter in modern times. Referring to the citations in Baily for the history of the question subsequent to 1680, I give what I have found in previous time; and I add a few new formulæ for consideration. The problem is, I doubt not, of infrequent occurrence. It never occurred to myself to want more than a look at the tables would furnish; but it may be suspected that, as the requirements of business become closer, inverse problems will acquire more importance than they now possess.

Given the term, and the present value or the amount, of an annuity, immediate or deferred, required the rate of interest. The earliest notice I find of any case of this question is in the work* of Dr. John Newton, whose writings, announced or referred to as those of "I. Newton," sometimes puzzle indifferent chronologists by their dates. It was believed and stated, at the time, that all that was new in this book was an unacknowledged communication from Michael Dary, presently spoken of. The problem—described (p. 103) as "one of which Mr. Brigges hath made no mention"—of the rate to a given term and present value, is solved by common interpolation, on a seven years' instance, at five years' purchase, with an answer 9·2107 per cent.

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* *The Scale of Interest;* or, the use of decimal Fractions, and the Table of Logarithmes, in the most easy and exact Resolving of all Questions in Anatocism, or Compound Interest, ... together with their use in the measuring of board, timber, ... for the use of the intended English Mathematical and Grammar School at Ross in

The next who attempted the problem is Michael Dary, a writer who deserves to be rescued from the utter oblivion into which he has fallen, for which the publication of the *Macclesfield Correspondence* furnishes the means (i. 99, 179, 204, 220; ii. 16, 166, 176, 200, 239, 272, 296, 361, 365, 367, 369, 454, 456, 473, 479, 556). These references are rendered necessary by the *Macclesfield Correspondence* having, as yet, no index whatever, no table of contents, and not even a list of the writers.

Michael Dary turns up in 1663 as a correspondent of Collins, being then a gauger at Bristol, by Collins's interest, as he intimates. He appears with a ready knowledge of the higher mathematics—of indivisibles, for example. By 1673, we find him in London, "very poor and void of employment"; and then seeking, without success, for what appears to have been the mastership of the Royal Mathematical School at Christ's Hospital, with a testimonial from Newton, and the good offices of Collins. But he had obtained, a month or two before, the post of gunner in the Tower; and in 1677 he is described as a tobacco-cutter: Wallis calls him "Dary the tobacco-cutter, a knowing man in algebra." Among his patrons were Jonas Moore; Sherburne, the translator of Manilius; and George Wharton, the royalist astrologer. Among his correspondents were Collins, James Gregory, and Newton, from all of whom letters to him are preserved. There is also a letter from him to Collins in the Museum, published by the Historical Society of Science. After giving an equation, it runs as follows:—

> "Mr. Collins,—I have been lately trying to break biquadratique equations into two quadratique ones, and I have effected my purpose in a great many, some by the aliquote parts, and some by the cubicall mant, but this soure crabb I cannot deal with by no method . . . . Your servant,
>
> "MICHAEL DARY.
>
> "*Tower, the 8th February*, 1674-5."

Herefordshire. Also, a direction for Deans and Chapters . . . . in their Letting of Leases and Taking of Fines. By John Newton, D.D., and one of His Majesty's Chaplains. . . . . London, printed for Dixy Page . . . and Allen Bancks . . . 1668. 8vo.

August 15, 1674, Dary forwarded to Newton the theorem, or a case of it, of which I shall presently speak, in a letter beginning—"Although I sent you three papers yesterday, I cannot refrain from sending you this. I have had fresh thoughts this morning . . . ." The letter ends with—"Sir, pray do not count me troublesome, for I could not forbear but send this by Stiles,* the carrier, who is paid for the carriage. Pray remember me about the series of logarithms." No wonder that Newton, two months afterwards, writing to Collins, should say, "Mr. Dary is very solicitous about mathematics."

Dary published two works—one of *Miscellanies*, before 1673, which I have never seen; the other,† the *Interest Epitomised*, which is now very rare. The work is described in the *Correspondence* as about to be printed at the author's expense, for want of "undertakers;" but, by the dedication, it appears that the expenses were paid by Jonas Moore, Sherburne, Wharton, and James Hoare, Jun.—the last the only one of the four whose name has not descended. This work contains the theorem, above described as sent to Newton, which is now one of considerable importance in the calculus of functions, though I am not aware that it has ever been attributed to its first discoverer. It is as follows:—If a certain function, ϕ, be used as follows, $b=\phi a$, $c=\phi b$, $d=\phi c$, &c., then, if ϕa, ϕb, ϕc, &c. do not increase without limit, but approach to either a finite or zero limit, that limit is a solution of the equation $x=\phi x$. Dary applies this, *inter alia*, to the question of finding the rate of interest, as follows: If $ar=1-(1+r)^{-n}=1-v^n$, where n is the term of years, r the rate per pound, and a the purchase of the annuity of £1, then, if $(1-v^n)\div a$ be calculated by an approximate value of r, the result is a *nearer* value of r, which may be used in the same way to get a still nearer value; and so on. This is the first genuine solution; and, as the problem wants a name, I propose to call it *Dary's problem*. The approximation converges but slowly, and the method should only be used when an *inferior* value is commenced with, and it is desired to find a *superior* value also. Let r_1 be an inferior value of commencement, from which r_2 and r_3 are found by the method; then

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* John Stiles, the Cambridge carrier, is a person of note in the *Macclesfield Correspondence*. He takes the books backwards and forwards, returns with the money, and his name is a household word in the correspondence between Cambridge and London. He might have said to Newton, *Ubi tu Jupiter, ego Mercurius*.

† *Interest Epitomised, both Compound and Simple.* Very useful for every one that Lendeth or Borroweth; and for Purchasing and Selling of Annuities or Pensions, and Leases in Reversion. Whereunto is added, a Short Appendix for the Solution of *Adfected Equations* in Numbers by Approachment: performed by Logarithms. By Michael Dary, Philomath. London: Wm. Godbid, 1677. 8vo.

$r_3+\dfrac{(r_3-r_2)^2}{(r_2-r_1)-(r_3-r_2)}$ is a *superior* value, and nearer than r_3. The method converges more rapidly when we commence with a superior value; but the last formula does not then give an inferior value.

Newton himself gave a hint towards the solution of the problem, in reply to an inquiry of Collins, which seems to have been made on the part of Dary. In a letter to Collins (Feb. 6, 1669-70, *Macclesfield Correspondence*, vol. ii. p. 287), Newton writes as follows:—

> "To know at what rate (N per cent.) an annuity of B is purchased for 31 years at the price A.
>
> The rule is, $\dfrac{6 \text{ log. of } \frac{31B}{A}}{100-50 \text{ log. of } \frac{31B}{A}}=\text{log. of } \frac{100+N}{100}$."

Newton then works an example, $B=100$, $A=1200$, and deduces $N=7{\cdot}43$. He then proceeds as follows:—

> "This rule is not exact, but yet so exact as never to fail above 2*d.* or 3*d.* at most, when the rate is not above £16 per cent. And if the rate be above £16 or £18 per cent., or, which is all one, if $A<6B$, then this rule, $\dfrac{A+B}{A}=\dfrac{100+N}{100}$, will not err above 2*s.* You may try the truth of these rules by the equation $X^{32}=\dfrac{A+B}{A}X^{31}-\dfrac{B}{A}$, putting $\dfrac{100+N}{100}=X$, and working in logarithms."

In a subsequent letter (Feb. 18) Newton adds:—

> "The solution of the annuity problem, if it will be of any use, you have my leave to insert it into the *Philosophical Transactions*, so it be without my name to it. For I see not* what there is desirable in public esteem, were I able to acquire and maintain it. It would, perhaps, increase my acquaintance, the thing which I chiefly study to decline. Of that problem I could give exacter solutions, but that I have no leisure at present for computations."

I cannot find that this method was published: had it been in print, I cannot doubt that Halley, whose formula is closely connected with this, would have preferred a direct logarithmic form. If we take two terms of $2+t+\&c.$, in the formula presently given, remembering that $2+t=2(1-\frac{1}{2}t)^{-1}$ nearly when t is small, we have the *form* preferred by Newton. This will give

$$\text{log. }(1+r)=\frac{6(\text{log. } n-\text{log. } a)}{3(n+1)-\dfrac{n-1}{2}\,\dfrac{\text{log. } n-\text{log. } a}{\cdot 4342945}},$$

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* This is, I believe, the earliest manifestation now known of the temperament which afterwards put the world in some danger of never knowing Newton's discoveries.

which is the *form* of Newton's result, B being unity, and A being a. But if $n=31$, this gives 96, and 34·5 in lieu of 100 and 50. I suppose that Newton roughly amended the coefficients, so as to make the result more accurate for the high rates of interest then in use, and less accurate, relatively, for the lower ones. Perhaps, deriving the form from general considerations, as an approximation representing two terms of the series for log. $(1+r)$, he determined the coefficients 100 and 50 by comparison with tables or results. From the letters cited, it seems that Newton expanded the logarithm of a root of any binomial equation, and that this expansion was the "series of logarithms" about which Dary inquires.

Taking the hint of the form of expansion from Newton, I find the following results by common methods :—

Let r be the interest of £1 for one year, n the number of years of an annuity payable yearly, and a its present value. Given a and n, required r.

$$\text{We have, } a=\frac{1-(1+r)^{-n}}{r}. \quad \text{Let } t=\frac{1}{3}\,\frac{n-1}{n+1}\,\frac{\text{log. } n-\text{log. } a}{\cdot 4342945}\,;$$

$$\text{then, log. } (1+r)=\frac{\text{log. } n-\text{log. } a}{n+1}\cdot\left\{1+\frac{1}{1-t}-\frac{1}{20}\,\frac{t^3(1+t)}{(1-t)^2}\right\}.$$

Let A represent the accumulations of the same annuity at the moment after the last payment. We have

$$A=\frac{(1+r)^n-1}{r}. \quad \text{Let } t=\frac{1}{3}\,\frac{n+1}{n-1}\,\frac{\text{log. A}-\text{log. } n}{\cdot 4342945}\,;$$

$$\text{then, log. } (1+r)=\frac{\text{log. A}-\text{log. } n}{n-1}\cdot\left\{1+\frac{1}{1+t}+\frac{1}{20}\,\frac{t^3(1-t)}{(1+t)^2}\right\}.$$

The following formula is not calculated to the same extent :— Let the annuity be deferred for m years, let a be its present value. We have then—

$$a=\frac{(1+r)^{-m}-(1+r)^{-n}}{r}. \quad \text{Let } t=\frac{1}{3}\,\frac{n^2-1}{(2m+n+1)^2}\cdot\frac{\text{log. } n-\text{log. } a}{\cdot 4342945}\,;$$

$$\text{then, log. } (1+r)=\frac{\text{log. } n-\text{log. } a}{2m+n+1}\left\{1+\frac{1}{1-t}-\frac{1}{20}t^3\right\}.$$

I suspect that, as before, $\frac{t^3(1+t)}{(1-t)^2}$ would be more accurate than t^3; but I have not proved it.

I owe the following verifications to my friend Mr. Peter Gray; the rate is 5 per cent., and the values of a given by the common table are used, which of course ought to give $r=\cdot 05$:—

n.	Baily.	De Morgan.
10	·05000	·05000
25	·05003	·05001
50	·05019	·05004
75	·05060	·05014
100	·05140	·05019

At 10 per cent., $n=21$ gives ·1001, and $n=100$ gives ·1063. I doubt if any other formula of the kind yet given has borne the test of 10 per cent.

The following is Mr. Gray's way of arranging the calculation, $n=100$, $a=19\cdot84791$:—

Log. n	2·000000	
Log. a	1·297715	
Log. n—log. a	·702285	$\bar{1}$·846514
$n+1$		2·004321
(Log. n—log. a)$(n+1)^{-1}$		$\bar{3}$·842193*
$n-1$		1·995635
$(3\times\cdot4342945)^{-1}$		$\bar{1}$·885094
t	·528350	$\bar{1}$·722922
$1-t$	·471650	$\bar{1}$·673620
$(1-t)^{-1}$	2·12021	0·326380
t^3		$\bar{1}$·168766
$1+t$	1·528350	0·184223
$(1-t)^{-2}$		0·652760
20^{-1}		$\bar{2}$·698970
Product of last four	·05067	$\bar{2}$·704719
$1+(1-t)^{-1}$	3·12021	
	3·06954	0·487074
		$\bar{3}$·842193*
Log. $(1+r)$	·021344	
$1+r$	1·050185	$\bar{2}$·329267
True answer	1·050000	

In the formula for r from A, taking data which ought to give $r=\cdot05$, Mr. Gray finds ·05083, ·05001, ·04989, ·04993, ·04971, at 10, 25, 50, 75, 100 years. At $n=100$, when r should be ·1, it is ·09146.

The following direct approximations are not quite so good as the one which has been given. Let $z=\dfrac{2(n-a)}{n(n+1)}$,

$$r=z+\frac{n+2}{3}z^2+\frac{(n+2)(5n+7)}{36}z^3+\frac{(n+2)(17n^2+44n+29)}{270}z^4,$$

$$\frac{r}{1+r}=z+\frac{n-1}{3}z^2+\frac{(n-1)(5n-2)}{36}z^3+\frac{(n-1)(17n^2-10n+2)}{270}z^4.$$

At 10 per cent., and 100 years, these give $r=\cdot 09$. The second of these series, though very differently obtained, is a particular case of that given by Mr. James Meikle in this *Journal*, vol. v., p. 153. The co-efficients in the page cited, which are the common co-efficients obtained in the reversion of series, are quite correctly printed in the last two lines (the value of d): five more may be found, with a difference of notation easily reduced, in the article "Reversion of Series," in the *Penny Cyclopædia.*

All the preceding formulæ require no assistance from tables. The following will give, I think, better results than any I have seen in which an approximate answer is chosen directly from the table:—

Let a and n be as before, and let a' be the nearest to a in the tables, above or below; the rate per pound for a' being r, the approximate value. Take v^{n+1} and v^{n+2} from the table of present values, and calculate $(a'-a)r=p$, $a-nv^{n+1}=q$. Then calculate ρ from

$$\rho=\frac{p}{q+n\dfrac{n+1}{2}\dfrac{pv^{n+2}}{q}},$$

and $r+\rho$ is a much nearer approximation. I have put down the form which I find most easy to calculate; but every one must please himself in this respect. Taking $n=98$, $a=21\cdot 924788$, I find $a'=24\cdot 464607$, $r=\cdot 04$, and $\rho=\cdot 00498$, whence $r+\rho=\cdot 04498$; which should be $\cdot 045$. This is a severe test for the formula, both as to n and ρ. The second term of the denominator is generally small compared with the first: reject it, and the remaining formula is, in substance, that given in Mr. David Jones's work.

The following formula is formed by carrying the method of the preceding still further:—Let a', as before, be the table result at the rate r,

$$p=(a'-a)r,$$
$$q=a-nv^{n+1},$$
$$k=n(n+1)v^{n+1},$$
$$s=k+\frac{n+2}{3}q,$$
$$t=\frac{vp}{q^2},$$

$$\rho=\frac{p}{q}\left\{1-\frac{kt}{2}(1-st)\right\}.$$

Rate required $=r+\rho$.

When $n=100$, $a=9\cdot 5233704$, the result is $r=\cdot 10$, $r+\rho=\cdot 10499991$, and should be $\cdot 105$.